A Continuum Treatment of Growth in Tissue – Mass Transport Coupled with Mechanics

H. Narayanan, K. Garikipati, E. M. Arruda, K. Grosh, S. Calve
University of Michigan, Ann Arbor

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broad goals

- mathematical and computational models of the processes of tissue development
  - models that are physiologically appropriate and thermodynamically valid
  - quantitative model motivated and validated by experiment
- experiments on and characterization of *in vitro* engineered tissue
  - model drives the controlled experiments

development of biological tissue

distinct processes of tissue development: [taber - 1995]

- growth – addition/loss of mass
  - *densification of bone*
- remodelling – change in microstructure
  - *alignment of trabeculae of bones to axis of external loading*
- morphogenesis – change in macroscopic form
  - *development of an embryo from a fertilized egg*

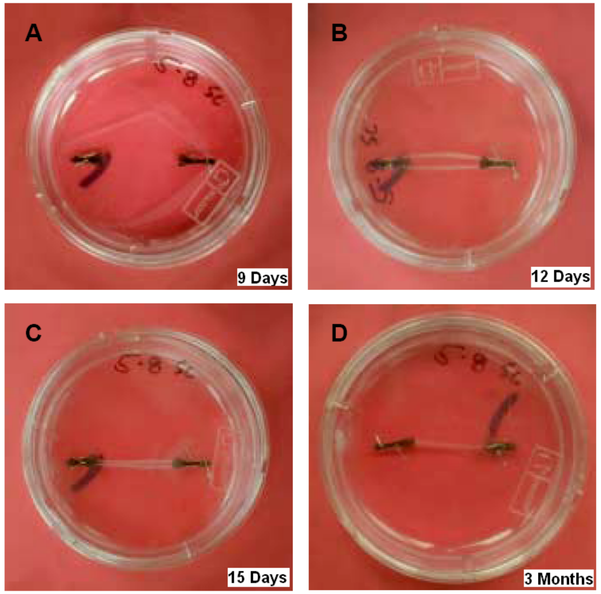
physics of growth

- open system with respect to mass
- interacting and interconverting species
- species diffusing with respect to a solid phase
  - *fluid, precursors, byproducts*
- mixture physics

our treatment involves the introduction of sources, sinks and fluxes of mass

biological model

engineered tissue *in vitro* that is morphologically and functionally similar to neonatal tissue: [calve et al., 2003]



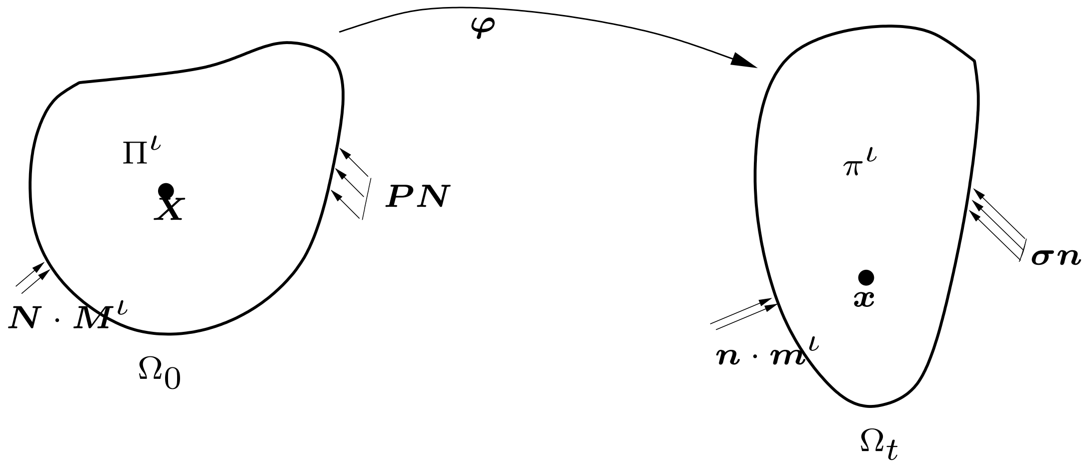
modelling background

- cowin and hegedus [1976]: solid tissue; mass source; irreversible sources of momentum and energy from perfusing fluid
- epstein and maugin [2000]: mass flux; irreversible fluxes of momentum and entropy
- kuhl and steinmann [2002]: configurational forces motivate mass flux

modelling of biological growth - this work

- multiple species undergoing transport, interconversion, mechanical and thermodynamic interactions
- other species deform with solid phase and diffuse with respect to it
- fully compatible with mixture theory
- detailed coupling of mechanics and mass balance
- thermodynamic consistency
- preliminary coupled computations

balance of mass



- tissue formed by reacting species – sources and sinks for species
- transport of precursors, fluid and byproducts – fluxes for species

balance of mass - equations

for a species ι, in local form, in Ω_0

$$\frac{\partial \rho_0^\iota}{\partial t} = \Pi^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{M}^\iota, \ \forall\, \iota = \alpha, \dots, \omega$$

the sources/sinks satisfy

$$\sum_{\iota=\alpha}^{\omega} \Pi^\iota = 0.$$

balance of mass - equations

for a species ι, in local form, in Ω_0

$$\frac{\partial \rho_0^\iota}{\partial t} = \Pi^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{M}^\iota, \; \forall\, \iota = \alpha, \ldots, \omega$$

for the solid phase

$$\frac{\partial \rho_0^s}{\partial t} = \Pi^s$$

ignoring short range motion of cells; e.g., during initial stages of wound healing

balance of mass - equations

for a species ι, in local form, in Ω_0

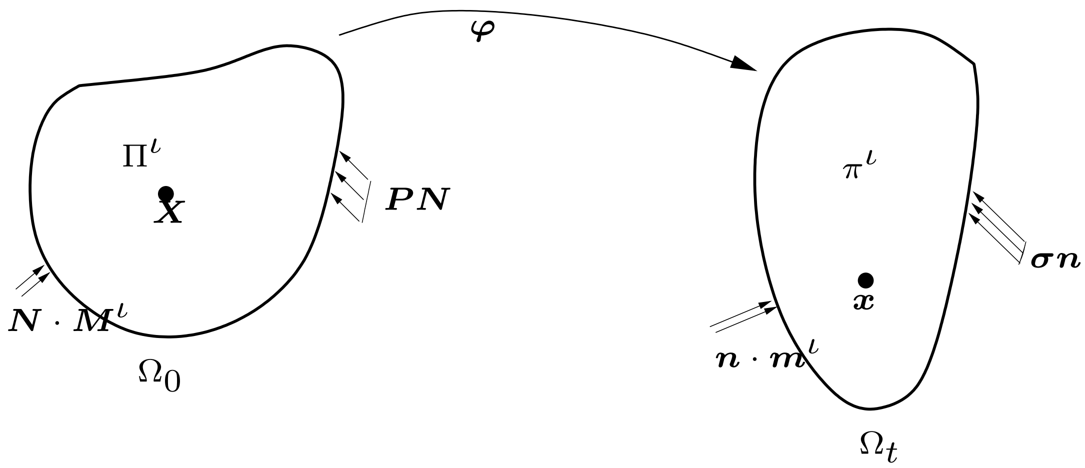
$$\frac{\partial \rho_0^\iota}{\partial t} = \Pi^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{M}^\iota, \; \forall\, \iota = \alpha, \ldots, \omega$$

for the fluid phase

$$\frac{\partial \rho_0^f}{\partial t} = -\boldsymbol{\nabla}_X \cdot \boldsymbol{M}^f$$

if sources for interstitial fluids are absent; e.g., no lymph glands

balance of linear momentum



- linear momentum balance coupled with mass transport – sources/sinks and fluxes contribute to the momenta
- material velocity relative to the solid $\boldsymbol{V}^\iota = (1/\rho_0^\iota)\boldsymbol{F}\boldsymbol{M}^\iota$

balance of linear momentum - equations

for a species ι, in local form, in Ω_0

$$\rho_0^\iota \frac{\partial}{\partial t}\left(\boldsymbol{V} + \boldsymbol{V}^\iota\right) = \rho_0^\iota \left(\boldsymbol{g} + \boldsymbol{q}^\iota\right) + \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - \left(\boldsymbol{\nabla}_X \left(\boldsymbol{V} + \boldsymbol{V}^\iota\right)\right)\boldsymbol{M}^\iota,\ \forall\, \iota = \alpha, \dots, \omega$$

balance of linear momentum - equations

for a species ι, in local form, in Ω_0

$$\rho_0^\iota \frac{\partial}{\partial t}(\boldsymbol{V} + \boldsymbol{V}^\iota) = \rho_0^\iota (\boldsymbol{g} + \boldsymbol{q}^\iota) + \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - (\boldsymbol{\nabla}_X (\boldsymbol{V} + \boldsymbol{V}^\iota))\, \boldsymbol{M}^\iota, \; \forall\, \iota = \alpha, \dots, \omega$$

balance of linear momentum - equations

for a species ι, in local form, in Ω_0

$$\rho_0^\iota \frac{\partial}{\partial t}\left(\boldsymbol{V} + \boldsymbol{V}^\iota\right) = \rho_0^\iota \left(\boldsymbol{g} + \boldsymbol{q}^\iota\right) + \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - \left(\boldsymbol{\nabla}_X \left(\boldsymbol{V} + \boldsymbol{V}^\iota\right)\right) \boldsymbol{M}^\iota, \ \forall\, \iota = \alpha, \ldots, \omega$$

balance of linear momentum - equations

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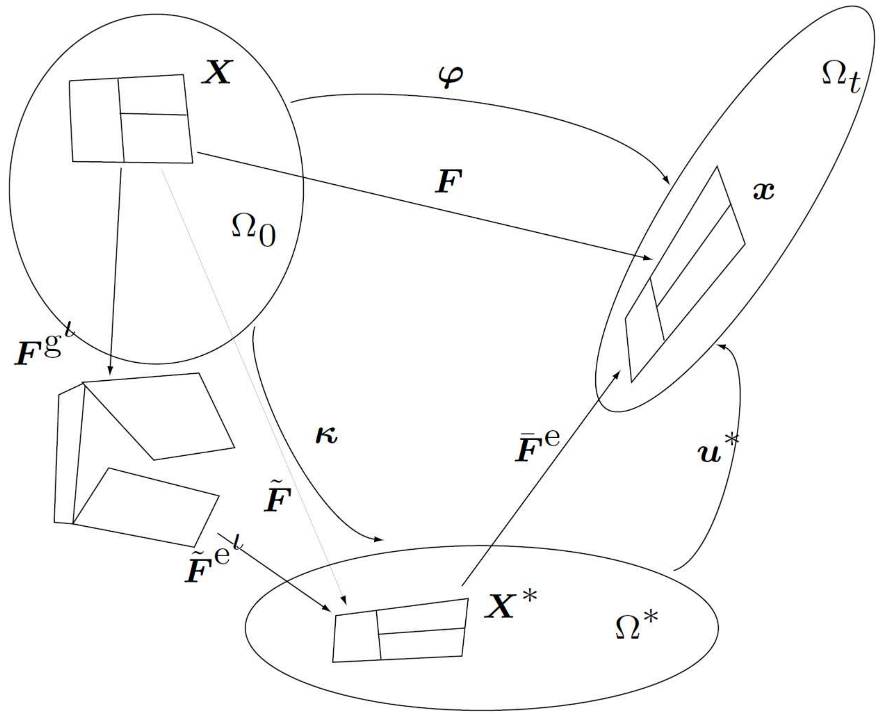
$$\rho_0^\iota \frac{\partial}{\partial t}\left(\boldsymbol{V} + \boldsymbol{V}^\iota\right) = \rho_0^\iota\left(\boldsymbol{g} + \boldsymbol{q}^\iota\right) + \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - \left(\boldsymbol{\nabla}_X\left(\boldsymbol{V} + \boldsymbol{V}^\iota\right)\right)\boldsymbol{M}^\iota,\ \forall\,\iota = \alpha, \dots, \omega$$

relation between mass sources Π^ι's and interaction forces $\boldsymbol{q}^\iota$'s,

$$\sum_{\iota=\alpha}^{\omega}\left(\rho_0^\iota \boldsymbol{q}^\iota + \Pi^\iota \boldsymbol{V}^\iota\right) = 0$$

kinematics of growth



kinematics of growth

$$\boldsymbol{F} = \bar{\boldsymbol{F}}^{\mathrm{e}} \tilde{\boldsymbol{F}}^{\mathrm{e}^{\iota}} \boldsymbol{F}^{\mathrm{g}^{\iota}}$$

- $\boldsymbol{F}^{\mathrm{g}^{\iota}}$ is a kinematic "growth" tensor , $\boldsymbol{F}^{\mathrm{e}^{\iota}} = \bar{\boldsymbol{F}}^{\mathrm{e}} \tilde{\boldsymbol{F}}^{\mathrm{e}^{\iota}}$ is the elastic deformation gradient

- residual stress due to $\tilde{\boldsymbol{F}}^{\mathrm{e}^{\iota}}$

energy, first law

balance of energy for a species ι, in local form, in Ω_0

$$\rho_0^\iota \frac{\partial e^\iota}{\partial t} = \boldsymbol{P}^\iota : \dot{\boldsymbol{F}} + \boldsymbol{P}^\iota : \boldsymbol{\nabla}_X \boldsymbol{V}^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{Q}^\iota + r_0^\iota + \rho_0^\iota \tilde{e}^\iota - \boldsymbol{\nabla}_X e^\iota \cdot (\boldsymbol{M}^\iota)$$

energy, first law

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$$\rho_0^\iota \frac{\partial e^\iota}{\partial t} = \boldsymbol{P}^\iota : \dot{\boldsymbol{F}} + \boldsymbol{P}^\iota : \boldsymbol{\nabla}_X \boldsymbol{V}^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{Q}^\iota + r_0^\iota + \rho_0^\iota \tilde{e}^\iota - \boldsymbol{\nabla}_X e^\iota \cdot (\boldsymbol{M}^\iota)$$

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energy, first law

balance of energy for a species ι, in local form, in Ω_0

$$\rho_0^\iota \frac{\partial e^\iota}{\partial t} = \boldsymbol{P}^\iota : \dot{\boldsymbol{F}} + \boldsymbol{P}^\iota : \boldsymbol{\nabla}_X \boldsymbol{V}^\iota - \boldsymbol{\nabla}_X \cdot \boldsymbol{Q}^\iota + r_0^\iota + \rho_0^\iota \tilde{e}^\iota - \boldsymbol{\nabla}_X e^\iota \cdot (\boldsymbol{M}^\iota)$$

where the interaction terms satisfy the relation,

$$\sum_{\iota=\alpha}^{\omega} \left(\rho_0^\iota \boldsymbol{q}^\iota \cdot (\boldsymbol{V} + \boldsymbol{V}^\iota) + \Pi^\iota \left(e^\iota + \frac{1}{2} \| \boldsymbol{V} + \boldsymbol{V}^\iota \|^2 \right) + \rho_0^\iota \tilde{e}^\iota \right) = 0$$

entropy, second law

$$\sum_{\iota=\alpha}^{\omega} \rho_0^\iota \frac{\partial \eta^\iota}{\partial t} \geq \sum_{\iota=\alpha}^{\omega} \left(\frac{r^\iota}{\theta} - \boldsymbol{\nabla}_X \eta^\iota \cdot \boldsymbol{M}^\iota - \frac{\boldsymbol{\nabla}_X \cdot \boldsymbol{Q}^\iota}{\theta} + \frac{\boldsymbol{\nabla}_X \theta \cdot \boldsymbol{Q}^\iota}{\theta^2} \right)$$

combine first and second laws to get the dissipation inequality

constitutive relations

constitutive hypothesis: $e^\iota = \hat{e}^\iota(\boldsymbol{F}^{\mathrm{e}^\iota}, \rho_0^\iota, \eta^\iota)$

constitutive relations consistent with the dissipation inequality:

$$\boldsymbol{P}^\iota = \rho_0^\iota \frac{\partial e^\iota}{\partial \boldsymbol{F}^{\mathrm{e}^\iota}}, \forall\, \iota \quad \circ \text{ hyperelastic material}$$

$$\theta = \frac{\partial e^\iota}{\partial \eta^\iota}, \ \forall\, \iota \quad \circ \text{ thermal physics}$$

$$\boldsymbol{Q}^\iota = -\boldsymbol{K}^\iota \boldsymbol{\nabla}_X \theta, \ \forall\, \iota \quad \circ \text{ fourier law}$$

$$\boldsymbol{u} \cdot \boldsymbol{K}^\iota \boldsymbol{u} \geq 0\, \forall \boldsymbol{u} \in \mathbb{R}^3 \quad \text{(semi-positive definite conductivity)}$$

constitutive relations

constitutive relation for flux of each transported species:

$$\boldsymbol{M}^{\iota} = \boldsymbol{D}^{\iota}\left(-\rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{g} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{\nabla}_X \cdot \boldsymbol{P}^{\iota} - \boldsymbol{\nabla}_X(e^{\iota} - \theta\eta^{\iota})\right)$$

$$\boldsymbol{u} \cdot \boldsymbol{D}^{\iota}\boldsymbol{u} \geq 0\, \forall \boldsymbol{u} \in \mathbb{R}^3$$

- $\boldsymbol{D}^{\iota}$ is the mobility

constitutive relations

constitutive relation for flux of each transported species:

$$\boldsymbol{M}^\iota = \boldsymbol{D}^\iota \left(-\rho_0^\iota \boldsymbol{F}^{\mathrm{T}} \frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^\iota \boldsymbol{F}^{\mathrm{T}} \boldsymbol{g} + \boldsymbol{F}^{\mathrm{T}} \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - \boldsymbol{\nabla}_X (e^\iota - \theta \eta^\iota) \right)$$

- driving force due to inertia

constitutive relations

constitutive relation for flux of each transported species:

$$\boldsymbol{M}^\iota = \boldsymbol{D}^\iota \left(-\rho_0^\iota \boldsymbol{F}^{\mathrm{T}} \frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^\iota \boldsymbol{F}^{\mathrm{T}} \boldsymbol{g} + \boldsymbol{F}^{\mathrm{T}} \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota - \boldsymbol{\nabla}_X (e^\iota - \theta \eta^\iota) \right)$$

- driving force due to gravity

constitutive relations

constitutive relation for flux of each transported species:

$$\boldsymbol{M}^{\iota} = \boldsymbol{D}^{\iota}\left(-\rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{g} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{\nabla}_X \cdot \boldsymbol{P}^{\iota} - \boldsymbol{\nabla}_X(e^{\iota} - \theta\eta^{\iota})\right)$$

- driving force due to stress gradient – darcy's law

constitutive relations

constitutive relation for flux of each transported species:

$$\boldsymbol{M}^{\iota} = \boldsymbol{D}^{\iota}\left(-\rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\frac{\partial \boldsymbol{V}}{\partial t} + \rho_0^{\iota}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{g} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{\nabla}_X \cdot \boldsymbol{P}^{\iota} - \boldsymbol{\nabla}_X(e^{\iota} - \theta\eta^{\iota})\right)$$

- driving force due to a chemical potential gradient

reduced dissipation inequality

with the constitutive relations ensuring the non-positiveness of certain terms the entropy inequality is reduced to

$$\mathcal{D} = \sum_{\iota=\alpha}^{\omega} \left(\rho_0^\iota \frac{\partial e^\iota}{\partial \rho_0^\iota} \frac{\partial \rho_0^\iota}{\partial t} - \boldsymbol{P}^\iota : \boldsymbol{\nabla}_X \boldsymbol{V}^\iota + \rho_0^\iota \boldsymbol{V}^\iota \cdot \left(\frac{\partial \boldsymbol{V}^\iota}{\partial t} + \left(\boldsymbol{\nabla}_X \boldsymbol{V}^\iota \right) \boldsymbol{F}^{-1} \boldsymbol{V}^\iota \right) \right)$$

$$+ \sum_{\iota=\alpha}^{\omega} \Pi^\iota \left(e^\iota + \frac{1}{2} \| \boldsymbol{V} + \boldsymbol{V}^\iota \|^2 \right)$$

$$+ \sum_{\iota=\alpha}^{\omega} \left(\rho_0^\iota \frac{\partial}{\partial t} \left(\boldsymbol{V} + \boldsymbol{V}^\iota \right) - \rho_0^\iota \boldsymbol{g} - \boldsymbol{\nabla}_X \cdot \boldsymbol{P}^\iota + \boldsymbol{\nabla}_X \left(\boldsymbol{V} + \boldsymbol{V}^\iota \right) \left(\rho_0^\iota \boldsymbol{F}^{-1} \boldsymbol{V}^\iota \right) \right) \cdot \boldsymbol{V} \leq 0$$

preliminary coupled computations

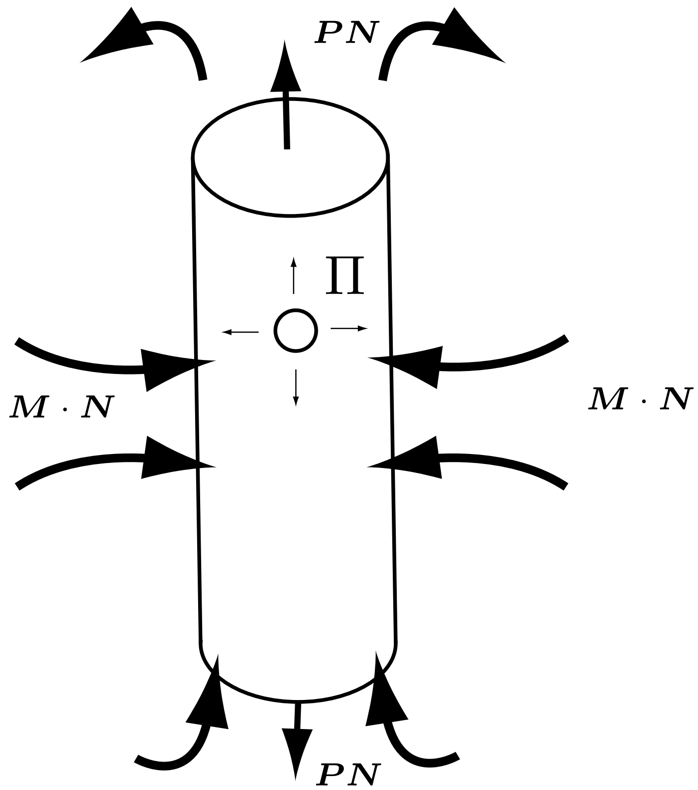
- biphasic model
  - worm-like chain model for collagen
  - nearly incompressible interstitial fluid with bulk compressibility of water, $\boldsymbol{\kappa}^{\mathrm{f}} = 2.25$ GPa
- fluid mobility $\boldsymbol{D}^{\iota}$ from swartz et al. [1999]
- "artificial" sources:

$$\Pi^{\mathrm{f}} = k^{\mathrm{f}}(\rho_0^{\mathrm{f}} - \rho_{0_{\mathrm{ini}}}^{\mathrm{f}}), \quad \Pi^{\mathrm{s}} = -\Pi^{\mathrm{f}}$$

- entropy of mixing:

$$\eta^{\iota}_{\mathrm{mix}} = -\frac{k}{\mathcal{M}^{\iota}} \log \frac{\rho_0^{\iota}}{\rho_0}$$

preliminary coupled computations



preliminary coupled computations - evolution of fields

view stress gradient-driven flux

view gravity-driven flux. view inertia-driven flux

view concentration gradient-driven flux

view total flux

view stress

view fluid source

summary and further work

- physiologically consistent continuum formulation describing growth in an open system
- relevant driving forces arise from thermodynamics – coupling with mechanics
- consistent with mixture theory

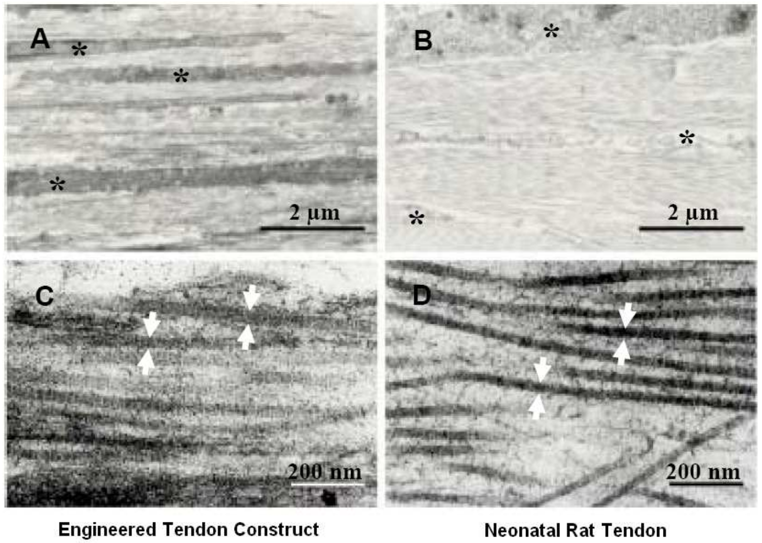
◦ formulated a theoretical framework for the remodelling problem

◦ engineering and characterization of growing, functional biological tissue

biological model - morphological comparison

morphological comparison of the engineered constructs to 2 day old neonatal rat tendon:

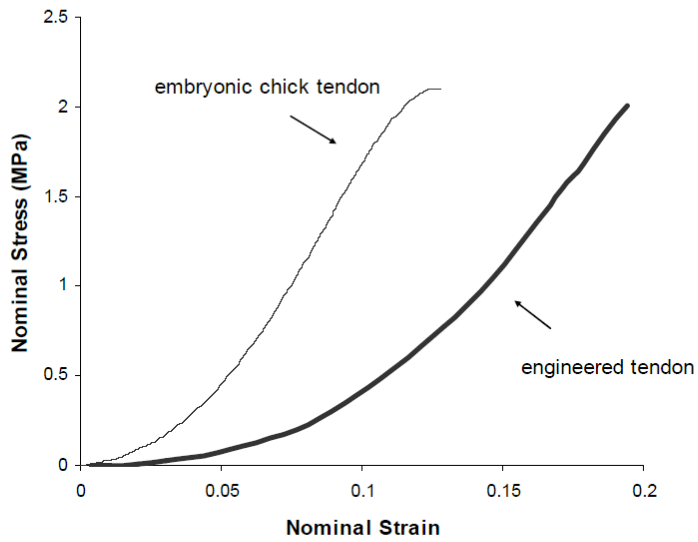
[calve et al., 2003]



biological model - mechanical comparison

comparison of the stress-strain response of the engineered construct to embryonic chicken tendon:

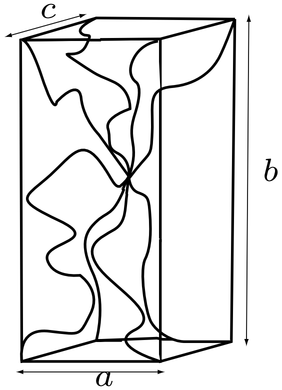
[calve et al., 2003]



cauchy stress

cauchy stress, $J^{\mathrm{e}^{\iota}} \boldsymbol{\sigma}^{\iota} = \boldsymbol{P}^{\iota} \boldsymbol{F}^{\mathrm{e}^{\iota T}}$, is symmetric

Worm-like chain model for solid collagen



$$\begin{aligned}\tilde{\rho}_0^{\mathrm{s}}\hat{e}^{\mathrm{s}}(\boldsymbol{F}^{\mathrm{e}^{\mathrm{s}}},\rho_0^{\mathrm{s}}) &= \frac{Nk\theta}{4A}\left(\frac{r^2}{2L}+\frac{L}{4(1-r/L)}-\frac{r}{4}\right)\\ &- \frac{Nk\theta}{4\sqrt{2L/A}}\left(\sqrt{\frac{2A}{L}}+\frac{1}{4(1-\sqrt{2A/L})}-\frac{1}{4}\right)\log(\lambda_1^{a^2}\lambda_2^{b^2}\lambda_3^{c}\\ &+ \frac{\gamma}{\beta}(J^{\mathrm{e}^{\iota}-2\beta}-1)+2\gamma\mathbf{1}\colon\boldsymbol{E}^{\mathrm{e}^{\mathrm{s}}}\end{aligned}$$

$$r=\frac{1}{2}\sqrt{a^2\lambda_1^{\mathrm{e}^2}+b^2\lambda_2^{\mathrm{e}^2}+c^2\lambda_3^{\mathrm{e}^2}},\quad \lambda_I^{\mathrm{e}}=\sqrt{\boldsymbol{N}_I\cdot\boldsymbol{C}^{\mathrm{e}}\boldsymbol{N}_I}$$

Mass Balance - Equations

For a species, in the integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0} \rho_0^\iota(\boldsymbol{X},t)\mathrm{d}V = \int\limits_{\Omega_0} \Pi^\iota(\boldsymbol{X},t)\mathrm{d}V - \int\limits_{\partial\Omega_0} \boldsymbol{M}^\iota(\boldsymbol{X},t)\cdot\boldsymbol{N}\mathrm{d}A,\ \forall\,\iota=\alpha,\dots,\omega \quad (1)$$

ρ_0^ι being the mass concentration of species ι and $\sum\limits_{\iota=\alpha}^{\omega}\rho_0^\iota = \rho_0$

The sources/sinks satisfy

$$\sum_{\iota=\alpha}^{\omega}\Pi^\iota = 0. \quad (2)$$

Balance of Linear Momentum - Equations

For a species ι, in the integral form written in Ω_0 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0} \rho_0^\iota(\boldsymbol{V}+\boldsymbol{V}^\iota)\mathrm{d}V = \int\limits_{\Omega_0} \rho_0^\iota \boldsymbol{g}\mathrm{d}V + \int\limits_{\Omega_0} \rho_0^\iota \boldsymbol{q}^\iota \mathrm{d}V + \int\limits_{\Omega_0} \Pi^\iota(\boldsymbol{V}+\boldsymbol{V}^\iota)\mathrm{d}V + \int\limits_{\partial\Omega_0} \boldsymbol{S}^\iota \boldsymbol{N}\mathrm{d}A - \int\limits_{\partial\Omega_0} (\boldsymbol{V}+\boldsymbol{V}^\iota)\boldsymbol{M}^\iota \cdot \boldsymbol{N}\mathrm{d}A \qquad (3)$$

$$\boldsymbol{q}^\iota = \sum_{\vartheta=\alpha,\vartheta\neq\iota}^{\omega} \boldsymbol{q}^{\iota\vartheta} \qquad (4)$$

On application of balance of mass, in local form, for the entire system

$$\sum_{\iota=\alpha}^{\omega} \rho_0^{\iota} \frac{\partial}{\partial t} \left(\boldsymbol{V} + \boldsymbol{V}^{\iota} \right) = \sum_{\iota=\alpha}^{\omega} \rho_0^{\iota} \left(\boldsymbol{g} + \boldsymbol{q}^{\iota} \right) + \sum_{\iota=\alpha}^{\omega} \boldsymbol{\nabla}_X \cdot \boldsymbol{S}^{\iota} - \sum_{\iota=\alpha}^{\omega} \left(\boldsymbol{\nabla}_X \left(\boldsymbol{V} + \boldsymbol{V}^{\iota} \right) \right) \boldsymbol{M}^{\iota} \tag{5}$$

Relation between Π^{ι}'s and $\boldsymbol{q}^{\iota}$'s,

$$\sum_{\iota=\alpha}^{\omega} \left(\rho_0^{\iota} \boldsymbol{q}^{\iota} + \Pi^{\iota} \boldsymbol{V}^{\iota} \right) = 0 \tag{6}$$

Balance of Angular Momentum - Equations

- In a purely mechanical theory, balance of angular momentum implies $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$.

- For a single species ι, in integral form in Ω_0,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0} \boldsymbol{\varphi}\times\rho_0^\iota(\boldsymbol{V}+\boldsymbol{V}^\iota)\mathrm{d}V \quad &= \quad \int\limits_{\Omega_0} \boldsymbol{\varphi}\times\left[\rho_0^\iota\left(\boldsymbol{g}+\boldsymbol{q}^\iota\right)+\Pi^\iota\left(\boldsymbol{V}+\boldsymbol{V}^\iota\right)\right]\mathrm{d}V \\
&+ \int\limits_{\partial\Omega_0} \boldsymbol{\varphi}\times\left(\boldsymbol{S}^\iota-\left(\boldsymbol{V}+\boldsymbol{V}^\iota\right)\otimes\boldsymbol{M}^\iota\right)\boldsymbol{N}\mathrm{d}A
\end{aligned} \tag{7}
$$

On simplification,

$$\int_{\Omega_0} \boldsymbol{V} \times \rho_0^\iota \boldsymbol{V}^\iota \mathrm{d}V \quad = \quad - \int_{\Omega_0} \boldsymbol{\epsilon} \colon \left(\left(\boldsymbol{S}^\iota - (\boldsymbol{V} + \boldsymbol{V}^\iota) \otimes \underbrace{\boldsymbol{M}^\iota}_{\rho_0^\iota \boldsymbol{F}^{-1} \boldsymbol{V}^\iota} \right) \boldsymbol{F}^{\mathrm{T}} \right) \mathrm{d}V \qquad (8)$$

On localizing,

$$\left(\boldsymbol{S}^\iota - \boldsymbol{V}^\iota \otimes \rho_0^\iota \boldsymbol{F}^{-1} \boldsymbol{V}^\iota \right) \boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F} \left(\boldsymbol{S}^\iota - \boldsymbol{V}^\iota \otimes \rho_0^\iota \boldsymbol{F}^{-1} \boldsymbol{V}^\iota \right)^{\mathrm{T}} \qquad (9)$$

But, $(\boldsymbol{V}^\iota \otimes \boldsymbol{F}^{-1} \boldsymbol{V}^\iota) \boldsymbol{F}^{\mathrm{T}} = \boldsymbol{V}^\iota \otimes \boldsymbol{V}^\iota$, which implies the symmetry: $\boldsymbol{S}^\iota \boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F} (\boldsymbol{S}^\iota)^{\mathrm{T}}$

This implies the partial Cauchy stresses are symmetric: $\boldsymbol{\sigma}^\iota = (\boldsymbol{\sigma}^\iota)^{\mathrm{T}}$

Balance of Energy - Equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_0} \rho_0^\iota \left(e^\iota + \frac{1}{2}\|\boldsymbol{V}+\boldsymbol{V}^\iota\|^2\right)\mathrm{d}V = \int\limits_{\Omega_0} \left(\rho_0^\iota \boldsymbol{g}\cdot(\boldsymbol{V}+\boldsymbol{V}^\iota) + r_0^\iota\right)\mathrm{d}V \\
+ \int\limits_{\Omega_0} \rho_0^\iota \boldsymbol{q}^\iota \cdot (\boldsymbol{V}+\boldsymbol{V}^\iota)\mathrm{d}V \\
+ \int\limits_{\Omega_0} \left(\Pi^\iota\left(e^\iota + \frac{1}{2}\|\boldsymbol{V}+\boldsymbol{V}^\iota\|^2\right) + \rho_0^\iota \tilde{e}^\iota\right)\mathrm{d}V \\
+ \int\limits_{\partial\Omega_0} \left((\boldsymbol{V}+\boldsymbol{V}^\iota)\cdot\boldsymbol{S}^\iota - \boldsymbol{M}^\iota\left(e^\iota + \frac{1}{2}\|\boldsymbol{V}+\boldsymbol{V}^\iota\|^2\right) - \boldsymbol{Q}^\iota\right)\cdot\boldsymbol{N}\mathrm{d}A. \qquad (10)
\end{aligned}
$$

On simplification localizing, and summing over all ι,

$$\sum_{\iota=\alpha}^{\omega} \rho_0^{\iota} \frac{\partial e^{\iota}}{\partial t} = \sum_{\iota=\alpha}^{\omega} \left(\boldsymbol{S}^{\iota} : \dot{\boldsymbol{F}} + \boldsymbol{S}^{\iota} : \boldsymbol{\nabla}_X \boldsymbol{V}^{\iota} - \boldsymbol{\nabla}_X \cdot \boldsymbol{Q}^{\iota} + r_0^{\iota} + \rho_0^{\iota} \tilde{e}^{\iota} \right) - \sum_{\iota=\alpha}^{\omega} \boldsymbol{\nabla}_X e^{\iota} \cdot (\boldsymbol{M}^{\iota}) \tag{11}$$

Where $\tilde{e}^{\iota}$ satisfies the relation,

$$\sum_{\iota=\alpha}^{\omega} \left(\rho_0^{\iota} \boldsymbol{q}^{\iota} \cdot (\boldsymbol{V} + \boldsymbol{V}^{\iota}) + \Pi^{\iota} \left(e^{\iota} + \frac{1}{2} \| \boldsymbol{V} + \boldsymbol{V}^{\iota} \|^2 \right) + \rho_0^{\iota} \tilde{e}^{\iota} \right) = 0 \tag{12}$$

The different terms - Mechanics

In the reference configuration Ω_0,

Π^ι is the source/sink term for species ι
$\boldsymbol{M}^\iota$ is the mass flux term for species ι
$\boldsymbol{S}^\iota$ is the partial first Piola-Kirchhoff stress on species ι
$\boldsymbol{N}$ is the outward normal at the surface
$\boldsymbol{g}$ is the body force acting on the entire system

The different terms - Mechanics

In the current configuration Ω_t,

π^ι is the source/sink term for species ι
$\boldsymbol{m}^\iota$ is the mass flux term for species ι
$\boldsymbol{\sigma}^\iota$ is the partial Cauchy stress on species ι
$\boldsymbol{n}$ is the outward normal at the surface
$\boldsymbol{g}$ is the body force acting on the entire system

The different terms - Mechanics

$\boldsymbol{V}$ is the velocity of the solid phase
$\boldsymbol{V}^\iota$ is the material velocity relative to the solid phase defined as $\boldsymbol{V}^\iota = (1/\rho_0^\iota)\boldsymbol{F}\boldsymbol{M}^\iota$
$\boldsymbol{q}^\iota$ is the net force exerted on species ι by all other species in the system

The different terms - Energy

e^{ι} is the internal energy of each species ι
$\boldsymbol{F}$ is the deformation gradient
$\boldsymbol{Q}^{\iota}$ is the heat flux term for species ι
r_0^{ι} is the heat supplied to species ι per unit reference volume
$\tilde{e}$ is the internal energy transferred to species ι from all other species